Extensions of Pairs of Linear Transformations Between Infinite-Dimensional Vector Spaces*

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Submitted by Hans Schneider

ABSTRACT

To a pair $A,B:V\to W$ of linear maps between complex vector spaces attach the pair (V,W) endowed with the operation $(\alpha,\beta)v=(\alpha A+\beta B)(v)$, $\alpha,\beta\in\mathbf{C}$, $v\in V$. A concept of rank, similar to the torsion-free rank of abelian groups, is definable for the systems (V,W). With appropriate morphisms, the systems from an abelian category and Ext^1 can be construed as a vector space valued functor. We find all the cases in which $\mathrm{Ext}^1((V,W),(X,Y))$, with $(X,Y),(V,W)$ indecomposable systems of rank 0 or 1, is finite-dimensional, and compute its dimension in these cases. This extends a former computation for finite-dimensional systems.

INTRODUCTION

Let $\mathbf{C}^2$ denote the complex vector space of pairs of complex numbers. A $\mathbf{C}^2$-*system* (V,W) is a pair of complex vector spaces V and W together with a *system operation* which is a $\mathbf{C}$-bilinear map $(e,v)\mapsto ev$ of $\mathbf{C}^2\times V$ into W. $\mathbf{C}^2$-systems are a tool in studying pairs of linear transformations. Indeed, given an ordered basis (a,b) of $\mathbf{C}^2$, the system operation of (V,W) is specified by the pair of linear transformations $v\mapsto av$, $v\mapsto bv$ of V into W. Finite sequences of N linear transformations between two complex vector spaces give rise to $\mathbf{C}^N$-systems, which are defined similarly. However, since in this paper we consider only the case $N=2$, we shall refer to a $\mathbf{C}^2$-system simply as a *system*.

*Research supported by the National Research Council of Canada.
†The contribution of this author forms part of his doctoral thesis submitted to Queen's University in October 1975.

A *homomorphism* of a system (X, Y) into a system (V, W) is a pair (ϕ, ψ) of linear transformations $\phi : X \to V$, $\psi : Y \to W$ such that

$$e\phi(x) = \psi(ex)$$

for all $e \in \mathbf{C}^2$, $x \in X$. With homomorphisms composed componentwise, the systems form a category equivalent to the category of all right modules over a hereditary subring of the ring of 3-by-3 matrices over the complex numbers $\mathbf{C}$ (see [3] or the introduction of [4]). For $\alpha \in \mathbf{C}$ let $\alpha_{(X,Y)}$ be the endomorphism of (X, Y) consisting of multiplications by α in X and Y. Then putting $\alpha(\phi, \psi) = (\phi, \psi)\alpha_{(X,Y)} = \alpha_{(V,W)}(\phi, \psi)$, we endow $\mathrm{Hom}((X, Y), (V, W))$ with the structure of a complex vector space.

The usual concepts of module theory translate to the terminology of systems. E.g., a system (X, Y) is a *subsystem* of (V, W) in case $X \subset V$, $Y \subset W$ and the pair of inclusions is a homomorphism of systems. It is a *direct summand* of (V, W) provided there exists a subsystem (U, Z) of (V, W) such that $V = X + U$, $W = Y + Z$, $X \cap U = 0$, $Y \cap Z = 0$. We refer to [1], [2] for further definitions involving systems. (These papers use "spectral" for "direct summand".)

As can be seen in the abovementioned references, in examining the structure of systems it is essential to know when all short exact sequences

$$E : 0 \to (V^1, W^1) \to (U, Z) \to (V^2, W^2) \to 0$$

with ends of given isomorphism types split; i.e., $\mathrm{Ext}\big((V^2, W^2), (V^1, W^1)\big) = 0$. (We write 0 for systems of the form $(0, 0)$, and since Ext^n vanishes identically for $n > 1$, we abbreviate Ext^1 to Ext.) If $\mathrm{cls}(E)$ is the congruence class of the extension E and $\alpha \in \mathbf{C}$, we put $\alpha\,\mathrm{cls}(E) = \mathrm{cls}(\alpha_{(V^1, W^1)}E)$. This endows $\mathrm{Ext}\big((V^2, W^2), (V^1, W^1)\big)$ with the structure of a complex vector space, Ext becomes a bifunctor to the category of such vector spaces, and all the maps in the associated long exact sequences are $\mathbf{C}$-linear (See e.g., [5, Chapter 3] or [3]).

While many cases of vanishing Ext are contained in [1] and [2], it is of interest to include them in a systematic computation of $\dim \mathrm{Ext}\big((V^2, W^2), (V^1, W^1)\big)$. This was carried out in [3] for arguments which are finite-dimensional $[\dim(V, W) = \dim V + \dim W]$ by means of a formula valid for $\mathbf{C}^N$-systems. The results for finite-dimensional $\mathbf{C}^2$-systems which are indecomposable (and this suffices due to the additivity of Ext) are recapitulated in Table 1 at the end of the paper. Our aim here is to extend this computation to all indecomposable systems which are of (torsion-free) rank not exceeding one. These are either torsion systems or torsion-free of rank one. We assume familiarity with the indecomposable finite-dimensional and torsion systems, their spanning chains and the designation of their isomorphism types. In these matters we follow [1]. We use also the classification of torsion-free systems of rank one by means of equivalence classes of height functions [2]. Since the main interest is in finding when the dimension is zero, we do not insist on determining the cardinality if it is infinite—we write $\dim \operatorname{Ext}((V^2, W^2), (V^1, W^1)) = \infty$ in such cases.

To avoid repetition, we make full use of former results. These, either by direct quotation or combined with considerations using the long exact sequences for Ext, enable us to complete all but one entry in the table without resorting to resolutions. We show that in the remaining case, in which the arguments of Ext are torsion-free of rank one and at least one is of infinite dimension, the required dimension is either 0 or ∞. We distinguish between the possibilities by a condition on the height functions. A partial result, Lemma 2.5, is used in finding the systems which are both purely simple and pure injective [6].

1. EXTENSIONS OF INDECOMPOSABLE TORSION SYSTEMS

According to [1, Corollary 9.16(b)] these are the systems of the types I^m, II_η^m and II_η^∞. We have to consider their extensions by systems which are of type $\mathrm{II}_\theta^\infty$ or are torsion-free of rank 1. Here m is a positive integer, η and θ belong to the extended complex plane $\tilde{\mathbf{C}}$, and the use of these parameters in the designation of isomorphism types is relative to a choice of an ordered basis (a,b) of $\mathbf{C}^2$ (see [1, pp. 281–282]).

Systems of the types I^m and II_η^∞ are divisible [1, Def. 7.1, Props. 7.2(b), 8.4]. According to [1, Lemma 7.3] a short exact sequence

$$0 \to (V^1, W^1) \to (U, Z) \to (V^2, W^2) \to 0$$

splits if (V^1, W^1) is divisible and (V^2, W^2) is spanned by one of certain kinds of proper chains. As systems of the types II_θ^n and $\mathrm{II}_\theta^\infty$ are spanned by such chains, we have

PROPOSITION 1.1.

(a) $\operatorname{Ext}(\mathrm{II}_\theta^\infty, \mathrm{I}^m) = 0$;
(b) $\operatorname{Ext}(\mathrm{II}_\theta^n, \mathrm{II}_\eta^\infty) = 0$;
(c) $\operatorname{Ext}(\mathrm{II}_\theta^\infty, \mathrm{II}_\eta^\infty) = 0$.

Here and elsewhere we abuse notation and replace symbols for systems by the symbols for their isomorphism types.

We often use the following well-known long exact sequences for Ext and the ensuing dimension inequalities. The sequences terminate on the right with 0 because, as we have already remarked, Ext^2 vanishes identically for systems.

LEMMA 1.2. *Let* $0 \to (V^1, W^1) \to (U, Z) \to (V^2, W^2) \to 0$ *be an exact sequence of systems, and let* (V, W) *be any system. Then we have the following exact sequences of vector spaces*:

(a)
$$0 \to \mathrm{Hom}((V^2, W^2), (V, W)) \to \mathrm{Hom}((U, Z), (V, W))$$
$$\to \mathrm{Hom}((V^1, W^1), (V, W))$$
$$\to \mathrm{Ext}((V^2, W^2), (V, W)) \to \mathrm{Ext}((U, Z), (V, W))$$
$$\to \mathrm{Ext}((V^1, W^1), (V, W)) \to 0;$$

(b)
$$0 \to \mathrm{Hom}((V, W), (V^1, W^1)) \to \mathrm{Hom}((V, W), (U, Z))$$
$$\to \mathrm{Hom}((V, W), (V^2, W^2))$$
$$\to \mathrm{Ext}((V, W), (V^1, W^1)) \to \mathrm{Ext}((V, W), (U, Z))$$
$$\to \mathrm{Ext}((V, W), (V^2, W^2)) \to 0.$$

Consequently

(c) $\dim \mathrm{Ext}((U, Z), (V, W)) \geqslant \dim \mathrm{Ext}((V^1, W^1), (V, W))$;
(d) $\dim \mathrm{Ext}((V, W), (U, Z)) \geqslant \dim \mathrm{Ext}((V, W), (V^2, W^2))$.

PROPOSITION 1.3.

(a) $\dim \mathrm{Ext}(\mathrm{II}_\eta^\infty, \mathrm{II}_\eta^m) = m$;
(b) $\mathrm{Ext}(\mathrm{II}_\theta^\infty, \mathrm{II}_\eta^m) = 0$ if $\theta \neq \eta$.

Proof.

(a) By an appropriate choice of the basis of $\mathbf{C}^2$ relative to which the parameter η is determined, we may assume that $\eta \neq \infty$. Call a suitable basis (a, b). We make use of the fact that the systems (X, Y) in which a acts as an isomorphism from X onto Y form a subcategory of the category of systems equivalent to the category of modules over the polynomial ring $\mathbf{C}[\zeta]$. To a $\mathbf{C}[\zeta]$-module X corresponds the system (X,X), where X is regarded as a complex vector space and the system operation is defined by $(\alpha a+\beta b)x=(\alpha+\beta\zeta)x$, $\alpha,\beta\in\mathbf{C}$, $x\in X$. To a module homomorphism $\phi: X\to V$ corresponds the homomorphism $(\phi,\phi):(X,X)\to(Y,Y)$ of systems.

Let U be the $\mathbf{C}[\zeta]$-module $\mathbf{C}[\zeta,(\zeta-\eta)^{-1}]/\mathbf{C}[\zeta]$. Consider the epimorphic endomorphism π of U defined by multiplication by $(\zeta-\eta)^m$, and let V^1 be its kernel. To the exact sequence of modules

$$0\to V^1\xrightarrow{\mu} U\xrightarrow{\pi} U\to 0$$

corresponds an exact sequence of systems

$$E: 0\to(V^1,V^1)\xrightarrow{(\mu,\mu)}(U,U)\xrightarrow{(\pi,\pi)}(U,U)\to 0.$$

Here (U,U) is of type II_η^∞ spanned by a chain $\big((u_k)_{-\infty}^{-1}(z_k)_{-\infty}^{0}\big)$ with $u_k=z_k=(\zeta-\eta)^k$, $k<0$ and $z_0=0$; while (V^1,V^1) is of type II_η^m. Lemma 1.2 (a) yields the exact sequence

$$\mathrm{Ext}\big((U,U),(V^1,V^1)\big)\xrightarrow{(\pi,\pi)^*}\mathrm{Ext}\big((U,U),(V^1,V^1)\big)$$
$$\to\mathrm{Ext}\big((V^1,V^1),(V^1,V^1)\big)\to 0.$$

It suffices to show that $(\pi,\pi)^*=0$ and hence $\mathrm{Ext}\big((U,U),(V^1,V^1)\big)\cong\mathrm{Ext}\big((V^1,V^1),\ (V^1,V^1)\big)$ since by Table 1 $\dim\mathrm{Ext}(\mathrm{II}_\eta^m,\ \mathrm{II}_\eta^m)=m$.

Consider an exact sequence

$$F: 0\to(V^1,V^1)\xrightarrow{(\kappa,\lambda)}(X,Y)\xrightarrow{(\sigma,\tau)}(U,U)\to 0$$

representing a congruence class of $\mathrm{Ext}\big((U,U),(V^1,V^1)\big)$. Let $\circ$ denote the system operation of (X,Y). We claim that the linear transformation T_a: $x\mapsto a\circ x$ is an isomorphism of X onto Y. We use the fact that a acts as the identity map in (V^1,V^1) and (U,U). If $y\in Y$, then there exists $x\in X$ such that $\tau(y)=\sigma(x)=a\sigma(x)=\tau(a\circ x)$. Hence for some $v_1\in V^1$ we have $y-a\circ x=\lambda(v_1)=\lambda(av_1)=a\circ\kappa(v_1)$, and $y=a\circ(x+\kappa v_1)$. This proves that $T_a(X)=Y$. Suppose $a\circ x=0$. Then $0=\tau(a\circ x)=a\sigma(x)=\sigma(x)$, i.e., $x\in\kappa(V^1)$. But on $\kappa(V^1)$ a acts injectively. Hence $x=0$, and T_a is injective. Define a system operation on (X,X) by $ex=T_a^{-1}(e\circ x)$, $x\in X$, $e\in \mathbf{C}^2$. In this system too, a acts as an identity map. It is now readily checked that we have a commutative diagram of systems

$$\begin{tikzcd}
F:0 \arrow[r] & (V^1,V^1) \arrow[r, "(\kappa,\lambda)"] \arrow[d, "(1,1)"'] & (X,Y) \arrow[r, "(\sigma,\tau)"] \arrow[d, "(1,T_a^{-1})"] & (U,U) \arrow[r] \arrow[d, "(1,1)"] & 0 \\
\bar{F}:0 \arrow[r] & (V^1,V^1) \arrow[r, "(\kappa,\kappa)"] & (X,X) \arrow[r, "(\sigma,\sigma)"] & (U,U) \arrow[r] & 0.
\end{tikzcd}$$

Moreover, if X is given the structure of a $\mathbf{C}[\zeta]$-module by requiring that $\zeta x=bx$, then κ and σ are module homomorphisms. It follows that we have a commutative diagram

$$\begin{tikzcd}
\bar{F}:0 \arrow[r] & (V^1,V^1) \arrow[r, "(\kappa,\kappa)"] \arrow[d] & (X,X) \arrow[r, "(\sigma,\sigma)"] \arrow[d] & (U,U) \arrow[r] \arrow[d, "(\pi,\pi)"] & 0 \\
\bar{F}:0 \arrow[r] & (V^1,V^1) \arrow[r, "(\kappa,\kappa)"] & (X,X) \arrow[r, "(\sigma,\sigma)"] & (U,U) \arrow[r] & 0,
\end{tikzcd}$$

where all the components of the vertical maps are multiplications by $(\zeta-\eta)^m$ on the appropriate modules. Since $\bar{F}$ is congruent to F and $(\zeta-\eta)^m V^1=0$, we get $(\pi,\pi)^*F\equiv \bar{F}(\pi,\pi)\equiv(0,0)\bar{F}\equiv 0$ as desired.

(b) Let (V,W) be a system of type $\mathrm{II}_\theta^\infty$ spanned by a chain

$$\begin{tikzcd}[column sep=small]
 & \cdots v_{-2} \arrow[dr, dash, "b_\theta"] & & v_{-1} \arrow[dr, dash, "b_\theta"] & \\
\cdots w_{-2} \arrow[ur, dash, "a"] & & w_{-1} \arrow[ur, dash, "a"] & & 0
\end{tikzcd},$$

where $b_\theta=b-\theta a$ if $\theta\in\mathbf{C}$, $b_\infty=a$. (If $\theta=\infty$, then (a,b) is replaced by (b,a) in defining the chain.) Using the exact sequence E of part (a), we have by Lemma 1.2(b) an exact sequence

$$\operatorname{Hom}\big((V,W),(U,U)\big)\to\operatorname{Ext}\big((V,W),(V^1,V^1)\big)\to\operatorname{Ext}\big((V,W),(U,U)\big).$$

If $(\phi,\psi)\in\operatorname{Hom}\big((V,W),(U,U)\big)$, then the chain

$$\begin{tikzcd}[column sep=small]
 & \cdots \phi(v_{-2}) \arrow[dr, dash, "b_\theta"] & & \phi(v_{-1}) \arrow[dr, dash, "b_\theta"] & \\
\cdots \psi(w_{-2}) \arrow[ur, dash, "a"] & & \psi(w_{-1}) \arrow[ur, dash, "a"] & & 0
\end{tikzcd}$$

spans the subsystem $(\phi(V),\psi(W))$ of (U,U). Since the only eigenvalue of (U,U) is η [1, Proposition 8.4] and $\theta\neq\eta$, we obtain inductively that $\phi(v_{-k})=0$, $\psi(w_{-k})=0$ for $k=1,2,\dots$. Thus $(\phi,\psi)=(0,0)$ and $\operatorname{Hom}((V,W),(U,U))=0$. Our statement now follows from Proposition 1.1(c). ■

PROPOSITION 1.4.

$$\dim \operatorname{Ext}(\mathrm{I}^n,\mathrm{II}_\eta^\infty)=\infty.$$

Proof. For every positive integer m we have, as in the proof of Proposition 1.3(a), an exact sequence

$$0\to\mathrm{II}_\eta^m\to\mathrm{II}_\eta^\infty\to\mathrm{II}_\eta^\infty\to 0.$$

Lemma 1.2(b) yields an exact sequence

$$\operatorname{Hom}(\mathrm{I}^n,\mathrm{II}_\eta^\infty)\to\operatorname{Ext}(\mathrm{I}^n,\mathrm{II}_\eta^m)\to\operatorname{Ext}(\mathrm{I}^n,\mathrm{II}_\eta^\infty).$$

We have $\operatorname{Hom}(\mathrm{I}^n,\mathrm{II}_\eta^\infty)=0$. The proof is similar to that just given for $\operatorname{Hom}(\mathrm{II}_\theta^\infty,\mathrm{II}_\eta^\infty)=0$ if $\theta\neq\eta$. Since by Table 1 $\dim\operatorname{Ext}(\mathrm{I}^n,\mathrm{II}_\eta^m)=m$ and m is arbitrary, it follows that $\operatorname{Ext}(\mathrm{I}^n,\mathrm{II}_\eta^\infty)$ contains subspaces of arbitrarily large dimension, as we wanted to show. ■

PROPOSITION 1.5. *If* (V^1,W^1) *is of one of the types* I^m, II_η^m *or* II_η^∞ *and* (V^2,W^2) *is torsion-free, then* $\operatorname{Ext}((V^2,W^2),(V^1,W^1))=0$.

Proof. In all cases (V^1,W^1) is a torsion system. Thus if

$$0\to(V^1,W^1)\xrightarrow{(\kappa,\lambda)}(U,Z)\to(V^2,W^2)\to 0$$

is an exact sequence, then $(\kappa,\lambda)(V^1,W^1)$ is the torsion part of (U,Z) and hence is pure in (U,Z). (See [1, Proposition 9.12], where the terms "eigenvalue part" and "quasi-spectral" are used instead of "torsion part" and "pure".) Since systems of types I^m, II_η^m are finite-dimensional and systems of type II_η^∞ are divisible, it follows from [1, Theorems 5.5, 9.15] that (V^1,W^1) is pure injective. Hence the exact sequence splits. ■

2. EXTENSIONS OF TORSION-FREE SYSTEMS OF RANK 1

We employ models of torsion-free systems of rank 1 with spaces which are subspaces of $\mathbf{C}(\zeta)$—the space of rational functions in an indeterminate ζ. A *height function* H is a function $\theta \mapsto H_\theta$ from $\tilde{\mathbf{C}}$ to the nonnegative integers and ∞. Fixing a basis (a,b) of $\mathbf{C}^2$, we denote by R^H the system (S,T) with spaces

$$S=\left[(\zeta-\theta)^{-k}:0<k<H_\theta+1, \theta\in\mathbf{C}\right]+\left[\zeta^k:0\leqslant k<H_\infty\right],$$

$$T=\left[(\zeta-\theta)^{-k}:0<k<H_\theta+1, \theta\in\mathbf{C}\right]+\left[\zeta^k:0\leqslant k<H_\infty+1\right],$$

and system operation defined by $(\alpha a+\beta b)f(\zeta)=(\alpha+\beta\zeta)f(\zeta)$, $\alpha,\beta\in\mathbf{C}$, $f(\zeta)\in S$. Here k ranges over integers, and the square brackets denote the subspace of $\mathbf{C}(\zeta)$ spanned by the indicated rational functions. Every torsion-free system of rank 1 is isomorphic to a system R^H for a suitable height function H. The systems R^H and $R^{H'}$ are isomorphic if and only if

(a) the set $\Delta=\{\theta\in\tilde{\mathbf{C}}:H_\theta\neq H'_\theta\}$ is finite and $H_\theta\neq\infty\neq H'_\theta$ for $\theta\in\Delta$;

(b) if one of the functions does not assume the value ∞, then $\sum_{\theta\in\Delta}H_\theta=\sum_{\theta\in\Delta}H'_\theta$.

If this is the case, we say that H and H' are *equivalent*.

For a given isomorphism type of torsion-free systems of rank 1, a change in the basis (a,b) results in a Moebius transformation in the domain of definition of the corresponding functions—the same transformation that affects the parameter θ in II_θ^m, $\mathrm{II}_\theta^\infty$ (see [1, p. 282]). Thus the conditions in the results below are invariant under such changes. For the above results see [2, Sec. 3].

In the sequel H^1,H^2 denote height functions and R^1,R^2 denote the systems R^{H^1},R^{H^2} (taken relative to a basis (a,b) which may be chosen conveniently).

The conditions in the following results will also involve, explicitly or implicitly, the sets

$$F^j=\{\theta\in\tilde{\mathbf{C}}:H^j_\theta<\infty\},$$

and

$$P^j=\{\theta\in\tilde{\mathbf{C}}:H_\theta>0\},\qquad j=1,2.$$

If we pass from the H^j's to equivalent height functions, the F^j's do not change, while P^2 changes by a finite set. Consequently, the conditions depend only on the isomorphism types of R^1, R^2 and not on the particular height functions used in describing them.

Proposition 2.1. *If R^1 is infinite-dimensional, then* $\dim \operatorname{Ext}(\mathrm{I}^n, R^1) = \infty$.

Proof. Note that R^1 is spanned by a "bouquet" of chains originating at 1 in the second space of this system. Corresponding to each $\theta \in \mathbf{C}$ with $H_\theta^1 > 0$ we have the chain

$$\begin{tikzcd}[column sep=small]
 & (\zeta-\theta)^{-H_\theta^1} \arrow[dl, dash, "a"'] \arrow[dr, dash, "b_\theta"] & \\
(\zeta-\theta)^{-H_\theta^1} & & (\zeta-\theta)^{-H_\theta^1+1}
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
\cdots & & (\zeta-\theta)^{-2} \arrow[dl, dash, "a"'] \arrow[dr, dash, "b_\theta"] & & (\zeta-\theta)^{-1} \arrow[dl, dash, "a"'] \arrow[dr, dash, "b_\theta"] & \\
\cdots & (\zeta-\theta)^{-2} & & (\zeta-\theta)^{-1} & & 1
\end{tikzcd}$$

if $H_\theta^1 < \infty$; while the chain proceeds indefinitely to the left if $H_\theta^1 = \infty$. Similarly, if $0 < H_\infty^1 < \infty$, the bouquet contain the chain

$$\begin{tikzcd}[column sep=small]
 & 1 \arrow[dl, dash, "a"'] \arrow[dr, dash, "b"] & & \zeta \arrow[dl, dash, "a"'] \arrow[dr, dash, "b"] & & \cdots & & \zeta^{H_\infty^1-1} \arrow[dl, dash, "a"'] \arrow[dr, dash, "b"] & \\
1 & & \zeta & & \zeta^2 & \cdots & \zeta^{H_\infty^1-1} & & \zeta^{H_\infty^1}
\end{tikzcd};$$

while for $H_\infty^1 = \infty$ the chain proceeds indefinitely to the right. Hence, we have

$$R^1/(0, \mathbf{C}\cdot 1) \cong \bigoplus_{\theta \in P^1} (X^\theta, Y^\theta),$$

where (X^θ, Y^θ) is of type $\mathrm{II}_\theta^{h_\theta}$, $h_\theta = H_\theta^1$. By Lemma 1.2(d), $\dim \operatorname{Ext}(\mathrm{I}^n, R^1) \geqslant \dim \operatorname{Ext}(\mathrm{I}^n, \oplus_{\theta \in P^1} \mathrm{II}^{h_\theta})$. Since R^1 is infinite-dimensional, either $h_\theta < \infty$ for every $\theta \in P^1$ and P^1 is infinite, or there exists $\eta \in P^1$ with $h_\eta = \infty$. In the first case, given any positive integer m, there exists a finite subset Q of P^1 such that $\Sigma_{\theta \in Q} h_\theta > m$. Then

$$\operatorname{Ext}\Big(\mathrm{I}^n, \bigoplus_{\theta \in P^1} \mathrm{II}_\theta^{h_\theta}\Big) \cong \operatorname{Ext}\Big(\mathrm{I}^n, \bigoplus_{\theta \in P^1 \setminus Q} \mathrm{II}_\theta^{h_\theta}\Big) \oplus \bigoplus_{\theta \in Q} \operatorname{Ext}\big(\mathrm{I}^n, \mathrm{II}_\theta^{h_\theta}\big).$$

By Table 1, $\dim \operatorname{Ext}(\mathrm{I}^n, \mathrm{II}_\theta^{h_\theta}) = h_\theta$. Therefore $\dim \operatorname{Ext}(\mathrm{I}^n, \oplus_{\theta \in P^1} \mathrm{II}_\theta^{h_\theta}) > m$, and the proposition follows. In the second case the present proposition results similarly from the additivity of Ext and Proposition 1.4 ■

PROPOSITION 2.2.

(a) *If* $H_\theta^1 = \infty$, *then* $\operatorname{Ext}(\mathrm{II}_\theta^n, R^1) = 0$.
(b) *If* $H_\theta^1 < \infty$, *then* $\dim \operatorname{Ext}(\mathrm{II}_\theta^n, R^1) = n$.

Proof. Denote by R the *full rational* system $R^H = (\mathbf{C}(\zeta), \mathbf{C}(\zeta))$ corresponding to the height function H which is identically ∞. The isomorphism type of R does not depend on the basis (a, b) of $\mathbf{C}^2$. From the remarks made at the beginning of the proof of Proposition 2.1 we see that $R/R^1 \cong \oplus_{\eta \in F^1} \mathrm{II}_\eta^\infty$. Thus, by Lemma 1.2(b), we have an exact sequence

$$\operatorname{Hom}(\mathrm{II}_\theta^n, R) \to \operatorname{Hom}\Big(\mathrm{II}_\theta^n, \bigoplus_{\eta \in F^1} \mathrm{II}_\eta^\infty\Big) \to \operatorname{Ext}\big(\mathrm{II}_\theta^n, R^1\big) \to \operatorname{Ext}(\mathrm{II}_\theta^n, R). \qquad (*)$$

A full rational system is divisible. Hence, by [1, Lemma 7.3], $\operatorname{Ext}(\mathrm{II}_\theta^n, R) = 0$.

(a) As in the proof of Proposition 1.3(b), one sees that for $\theta \neq \eta$, $\operatorname{Hom}(\mathrm{II}_\theta^n, \mathrm{II}_\eta^\infty) = 0$. Since the second term of (*) is isomorphic to a subspace of $\prod_{\eta \in F^1} \operatorname{Hom}(\mathrm{II}_\theta^n, \mathrm{II}_\eta^\infty)$, it vanishes too, and the statement is established. We observe that one could prove it directly as in [1, Lemma 7.3] because the assumption implies that the linear transformation corresponding to b_θ in R^1 is surjective.

(b) Since R is torsion-free, we have $\operatorname{Hom}(\mathrm{II}_\theta^n, R) = 0$. As the last term of (*) still vanishes, the two middle terms are isomorphic. In the present case $\theta \in F^1$, so that $\operatorname{Hom}(\mathrm{II}_\theta^n, \oplus_{\eta \in F^1} \mathrm{II}_\eta^\infty) \cong \operatorname{Hom}(\mathrm{II}_\theta^n, \mathrm{II}_\theta^\infty)$. To see that the dimension of the last space of homomorphisms in n, we may use the same models as in the proof of Proposition 1.3(a). Let U be the $\mathbf{C}[\zeta]$-module $\mathbf{C}[(\zeta - \theta)^{-1}]/\mathbf{C}[\zeta]$, and V^2 its submodule generated by $\overline{(\zeta - \theta)^{-n}}$, where the bar denotes taking the coset modulo $\mathbf{C}[\zeta]$. (Again there is no loss of generality in assuming $\theta \neq \infty$.) Then the corresponding systems (U, U) and (V^2, V^2) are of the types $\mathrm{II}_\theta^\infty$ and II_θ^n respectively. If $(\phi, \psi) \in \operatorname{Hom}\big((V^2, V^2), (U, U)\big)$, we conclude from $a\phi(v^2) = \psi(av^2)$, $v^2 \in V^2$ that $\phi = \psi$, and then from $b\phi(v^2) = \phi(bv^2)$ that ϕ commutes with multiplication by ζ; i.e., it is a homomorphism of $\mathbf{C}[\zeta]$-modules. Since conversely, for a module homomorphism ϕ, (ϕ, ϕ) is a homomorphism of systems, we have $\dim \operatorname{Hom}\big((V^2, V^2), (U, U)\big) = \dim \operatorname{Hom}(V^2, U)$. Suppose

$$\phi\big(\overline{(\zeta - \theta)^{-n}}\big) = \sum_k \alpha_k \overline{(\zeta - \theta)^{-k}},$$

where $\alpha_k \in \mathbf{C}$ and k ranges over a finite subset of the positive integers. Using the fact that multiplication by $(\zeta-\theta)^n$ annihilates $\overline{(\zeta-\theta)^{-n}}$, we see that $\alpha_k=0$ for $k>n$. Thus ϕ is the multiplication by $\Sigma_{k=1}^{n}\alpha_k(\zeta-\theta)^{n-k}$, and $\dim \operatorname{Hom}(V^2, U)=n$. ∎

PROPOSITION 2.3.

(a) *If* $H_\theta^1=\infty$, *then* $\operatorname{Ext}(\mathrm{II}_\theta^\infty, R^1)=0$.
(b) *If* $H_\theta^1<\infty$, *then* $\dim \operatorname{Ext}(\mathrm{II}_\theta^\infty, R^1)=\infty$.

Proof. The proof of part (a) is the same as that of Proposition 2.2(a) with ∞ replacing n throughout. Since a system of type $\mathrm{II}_\theta^\infty$ contains a subsystem of type II_θ^n for any positive integer n, part (b) follows from Proposition 2.2(b) and Lemma 1.2(c). ∎

REMARK 2.4. If R^1 is of type III^m, then the assumption of Proposition 2.3(b) is satisfied for every θ. Thus $\operatorname{Ext}(\mathrm{II}_\theta^\infty, \mathrm{III}^m)$ is infinite-dimensional.

LEMMA 2.5. *If there exists a* θ *in* $\tilde{\mathbf{C}}$ *such that* $H_\theta^1<\infty$ *and* $H_\theta^2=\infty$, *then* $\dim Ext(R^2, R^1)=\infty$.

Proof. By choosing the basis (a,b) of $\mathbf{C}^2$ appropriately we may assume $\theta=\infty$, i.e., $H_\infty^1<\infty$ and $H_\infty^2=\infty$. It is easy to see that there exists a height function H equivalent to H^1 such that $H_\infty=0$. Since $R^H\cong R^1$, there is no loss of generality in assuming that $H_\infty^1=0$. We make these assumptions for convenience.

Since $H_\infty^2=\infty$, R^2 contains the subsystem $P=(\mathbf{C}[\zeta], \mathbf{C}[\zeta])$. (The isomorphism type of P depends on the proportionality class $\{\alpha a : \alpha\in\mathbf{C}\}$.) By Lemma 1.2(c), it suffices to show that $\dim \operatorname{Ext}(P, R^1)=\infty$.

As in the proof of Proposition 2.2, we have an exact sequence

$$0\to R^1\to R \overset{(\sigma,\tau)}{\to} (V^2, W^2)\to 0$$

with σ,τ quotient maps. The subsystem $(\sigma,\tau)P$ of (V^2, W^2) is spanned by the infinite chain

$$\begin{tikzcd}[column sep=small]
 & \sigma(1) \arrow[dr, "b"] & & \sigma(\zeta) \arrow[dr, "b"] & & \sigma(\zeta^2)\cdots \arrow[dr, "b"] & \\
0 \arrow[ur, "a"] & & \tau(\zeta) \arrow[ur, "a"] & & \tau(\zeta^2) \arrow[ur, "a"] & & \tau(\zeta^3)\cdots.
\end{tikzcd}$$

Because $H^1_\infty=0$, the sequences $\left(\sigma(\zeta^k)\right)_{k=0}^\infty$, $\left(\tau(\zeta^k)\right)_{k=1}^\infty$ are linearly independent, and so $(\sigma,\tau)P$ is of type $\mathrm{II}_\infty^\infty$. This subsystem is a direct summand of (V^2,W^2), in conformity with the decomposition of (V^2,W^2) given in the proof of Proposition 2.2. Write $f\in\mathbf{C}(\zeta)$ in the form $f=p+q/r$, where p,q,r are polynomials and $\deg q<\deg r$, and put $\pi(\sigma(f))=\sigma(p)$, $\rho(\tau(f))=\tau(p)$. Since $H^1_\infty=0$, the "polynomial component" p of f is uniquely determined by $\sigma(f)$ [uniquely determined by $\tau(f)$ modulo a constant]. Note that $\tau(1)=0$. Thus π, ρ are well defined, and (π,ρ) is a projection of (V^2,W^2) onto $(\sigma,\tau)P$. We shall employ also the endomorphism (μ,ν) of $(\sigma,\tau)P$, where the linear maps μ, ν are defined by the requirements $\mu(\sigma(1))=0$, $\mu\left(\sigma(\zeta^k)\right)=\sigma(\zeta^{k-1})$, $\nu\left(\tau(\zeta^k)\right)=\tau(\zeta^{k-1})$ for $k\geqslant 1$.

The above exact sequence gives rise to an exact sequence

$$\mathrm{Hom}(P,R^1)\to\mathrm{Hom}(P,R)\xrightarrow{(\sigma,\tau)_*}\mathrm{Hom}\left(P,(V^2,W^2)\right)\to\mathrm{Ext}(P,R^1)$$
$$\to\mathrm{Ext}(P,R).$$

Every homomorphism of P into R^1 is a pair of multiplications by a rational function f (see the proof of [2, Theorem 3.6]). If we had $f\neq 0$, it would follow that the spaces of R^1 contain rational functions which have poles at ∞. This is excluded by the assumption $H^1_\infty=0$. Thus $f=0$, and $\mathrm{Hom}(P,R^1)=0$. By [1, Lemma 9.9] we have also $\mathrm{Ext}(P,R)=0$. Hence to prove the lemma it suffices to show that $(\sigma,\tau)_*\mathrm{Hom}(P,R)$ is of infinite codimension in $\mathrm{Hom}\left(P,(V^2,W^2)\right)$.

To every formal power series $g=\sum_{k=0}^\infty\alpha_k\zeta^k\in\mathbf{C}[[\zeta]]$ we attach a homomorphism $(\phi_g,\psi_g)\in\mathrm{Hom}(P,(V^2,W^2))$ as follows. We put $\phi_g(p)=\sum_{k=0}^\infty\alpha_k\mu^k\sigma(p)$, $\psi_g(p)=\sum_{k=0}^\infty\alpha_k\nu^k\tau(p)$ for $p\in\mathbf{C}[\zeta]$. The linear transformations ϕ_g, ψ_g are well defined because for every polynomial p, $\sigma(p)$ and $\tau(p)$ are annihilated by all sufficiently high powers of μ and ν respectively. The pair (ϕ_g,ψ_g) is easily seen to be a homomorphism from P to (V^2,W^2). As above, the general homomorphism in $\mathrm{Hom}(P,R)$ is of the form (χ_f,χ_f), where $\chi_f:\mathbf{C}[\zeta]\to\mathbf{C}(\zeta)$ is the multiplication by a rational function f. Thus if $(\phi_g,\psi_g)\in(\sigma,\tau)_*\mathrm{Hom}(P,R)$, then $(\phi_g,\psi_g)=(\sigma,\tau)(x_f,x_f)$ for some $f\in\mathbf{C}(\zeta)$. By the definition of (ϕ_g,ψ_g) its image is included in $(\sigma,\tau)P$. It follows that $(\phi_g,\psi_g)=(\pi,\rho)(\sigma,\tau)(\chi_f,\chi_f)$. If $f=s/t$, where $s,t\in\mathbf{C}[\zeta]$ and l is a positive integer larger than $\deg s-\deg t$, then the polynomial component of $f\zeta^l$ is divisible by ζ. Thus $(\pi\sigma\chi_f)(\zeta^l)=\pi\left(\sigma(f\zeta^l)\right)$ is a linear combination of the elements $\sigma(\zeta^k)$ with $k\geqslant 1$. On the other hand

$$\phi_g(\zeta^l)=\alpha_0\sigma(\zeta^l)+\alpha_1\sigma(\zeta^{l-1})+\cdots+\alpha_l\sigma(1).$$

It follows that $\alpha_l = 0$ for all l large enough, and g represents a polynomial. Since the map $g \mapsto (\phi_g, \psi_g)$ is linear and $\dim \mathbf{C}[[\zeta]]/\mathbf{C}[\zeta] = \infty$, we conclude that $\dim \operatorname{Hom}(P,(V^2,W^2))/(\sigma,\tau)_* \operatorname{Hom}(P,R) = \infty$. ∎

Lemma 2.6. *If*

$$F^1 \cap P^2 = \{\theta \in \tilde{\mathbf{C}} : H_\theta^1 < \infty \text{ and } H_\theta^2 > 0\}$$

is infinite, then $\dim \operatorname{Ext}(R^2, R^1) = \infty$.

Proof. If the set $(F^1 \cap P^2 \cap P^1)\backslash\{\infty\}$ is infinite, call it A. Otherwise, take A to be the infinite set $(F^1 \cap P^2)\backslash(P^1 \cup \{\infty\})$. Denote by R_A the system R^H, where H is the characteristic function of the subset A of $\tilde{\mathbf{C}}$. Since $A \subset P^2$, R_A is a subsystem of R^2, and it suffices by Lemma 1.2(c) to show that $\dim \operatorname{Ext}(R_A, R^1) = \infty$.

For each $\theta \in A$ let (U_θ, Z_θ) be a system of type III^2 spanned by a chain

$$\begin{tikzcd} & u_\theta \arrow[dl, dash, "a"'] \arrow[dr, dash, "b_\theta"] & \\ r_\theta & & z_\theta \end{tikzcd} .$$

It is easy to see that systems of type III^2 are projective [3]; hence so is $(U,Z) = \oplus_{\theta \in A}(U_\theta, Z_\theta)$. Identifying (U_θ, Z_θ) with the corresponding component of (U,Z), we define an epimorphism (σ,τ) of (U,Z) onto R_A by the requirements $\sigma(u_\theta) = \tau(r_\theta) = (\zeta - \theta)^{-1}$, $\tau(z_\theta) = 1$, $\theta \in A$. Fix an element η of A, and put $B = A\backslash\{\eta\}$. Then $\ker(\sigma,\tau) = (0,L)$, where L has the basis $(z_\theta - z_\eta)_{\theta \in B}$. Applying Lemma 1.2(a) to the exact sequence

$$0 \to (0,L) \overset{(0,\lambda)}{\to} (U,Z) \overset{(\sigma,\tau)}{\to} R_A \to 0,$$

where 0 and λ are canonical injections, what we have to show is that $\dim \operatorname{Hom}((0,L),R^1)/(0,\lambda)^* \operatorname{Hom}((U,Z),R^1) = \infty$.

To every function $g: B \to \mathbf{C}$ we attach a homomorphism $(0,\psi_g):(0,L) \to R^1$ by the requirements

$$\psi_g(z_\theta - z_\eta) = g(\theta)(\zeta - \theta)^{-h_\theta}, \qquad \theta \in B,$$

where $h_\theta = H_\theta^1$. We claim that if $(0,\psi_g)$ is in the image of $(0,\lambda)^*$ [namely, there exists $(\mu,\nu) \in \operatorname{Hom}((U,Z),R^1)$ with ν extending ψ_g], then g coincides with a rational function on a subset of B having a finite complement in B.

Indeed, we have for every $\theta \in B$

$$(\zeta-\theta)\,\mu(u_\theta)=b_\theta\mu(u_\theta)=\nu(b_\theta u_\theta)=\nu(z_\theta)=\nu(z_\eta)+\psi_g(z_\theta-z_\eta)$$

$$=\nu(z_\eta)+g(\theta)(\zeta-\theta)^{-h_\theta}.$$

Since the order of $\mu(u_\theta)$ at θ is at least $-h_\theta$, it follows that the Laurent expansion of $\nu(z_\eta)$ about θ is of the form

$$\nu(z_\eta)=-g(\theta)(\zeta-\theta)^{-h_\theta}+\sum_{k=-h_\theta+1}^{\infty}\alpha_{\theta k}(\zeta-\theta)^k.$$

If $F^1\cap P^2\cap P^1$ is infinite, then $h_\theta>0$ on B. Since $\nu(z_\eta)$ is a rational function, it has a finite set of poles in B, and $g(\theta)=0$ outside of this set. In case $(F^1\cap P^2)\backslash P^1$ is infinite, $h_\theta=0$ on B and $g(\theta)=-\nu(z_\eta)(\theta)$ for all $\theta\in B$. This proves our claim in both cases.

Since the map $g\mapsto(0,\psi_g)$ is linear, to show that $(0,\lambda)^*\operatorname{Hom}\big((U,Z),R^1\big)$ is of infinite codimension in $\operatorname{Hom}\big((0,L),R^1\big)$ it suffices to exhibit an infinite sequence $(g_j)_{j=1}^{\infty}$ of functions on B such that no nontrivial linear combination of the g_j's coincides with a rational function on a cofinal subset of B. One may, for instance, partition B into a sequence $(B_j)_{j=1}^{\infty}$ of infinite subsets, and take g_j to be the characteristic function of B_j. ∎

Corollary 2.7. *If R^2 is infinite-dimensional, then* $\dim\operatorname{Ext}(R^2,\mathrm{III}^m)=\infty$.

Proof. If $H^1_\infty=m-1$, $H^1_\theta=0$ for $\theta\in\mathbf{C}$, then R^1 is of type III^m. In this case $F^1=\tilde{\mathbf{C}}$. Since R^2 is infinite-dimensional, either there exists a θ in $\tilde{\mathbf{C}}$ with $H^2_\theta=\infty$ and Lemma 2.5 applies, or P^2 is infinite and the conclusion follows from Lemma 2.6.

Alternatively, we may use the fact that, according to Lemma 2.2 of [2], every finite-dimensional subsystem of R^2 is contained in a subsystem of some type III^n. By Lemma 1.2(c) and Table 1,

$$\dim\operatorname{Ext}(R^2,R^1)\geqslant\dim\operatorname{Ext}(\mathrm{III}^n,\mathrm{III}^m)=\max(0,n-m-1),$$

and n can be taken arbitrarily large here. ∎

LEMMA 2.8. *Suppose that*

(i) *At least one of the systems* R^1 *and* R^2 *is infinite-dimensional*;
(ii) $F^1 \subset F^2$;
(iii) $F^1 \cap P^2$ *is finite.*

Then $\mathrm{Ext}(R^2, R^1) = 0$.

Proof. Suppose first that $F^1 = \tilde{\mathbf{C}}$. Then by (iii) P^2 is finite and, since by (ii) $F^2 = \tilde{\mathbf{C}}$, H^2 is finitely valued on P^2. Thus $n = \sum_{\theta \in P^2} H_\theta^2 + 1$ is a positive integer, and R^2 is a finite-dimensional system of type III^n. By (i) R^1 is of infinite dimension. That in this case $\mathrm{Ext}(\mathrm{III}^n, R^1) = 0$ was shown in the proof of [2, Theorem 5.6].

We claim that if $F^1 \neq \tilde{\mathbf{C}}$, then H^2 is equivalent to a height function $\bar{H}^2$ such that with $\bar{P}^2 = \{\theta \in \mathbf{C} : \bar{H}_\theta^2 > 0\}$ we have $F^1 \cap \bar{P}^2 = \varnothing$. Indeed, if $F^2 \neq \tilde{\mathbf{C}}$ (namely, H^2 assumes the value ∞), we obtain $\bar{H}^2$ from H^2 by changing the values of H^2 to zero on $F^1 \cap P^2$. This is permissible because by (iii) the values of H^2 are changed on a finite set on which, by (ii), H^2 is finitely valued. If $F^2 = \tilde{\mathbf{C}}$, we have to preserve the sum of the values on the set on which H^2 is changed. Pick $\eta \in \tilde{\mathbf{C}} \setminus F^1$. Put

$$\bar{H}_\theta^2 = H_\theta^2 \qquad \text{for} \quad \theta \notin (F^1 \cap P^2) \cup \{\eta\},$$

$$\bar{H}_\theta^2 = 0 \qquad \text{for} \quad \theta \in F^1 \cap P^2;$$

$$\bar{H}_\eta^2 = H_\eta^2 + \sum_{\theta \in F^1 \cap P^2} H_\theta^2.$$

Noting that $\eta \in F^2$ and $\eta \notin F^1$, we see that $\bar{H}^2$ satisfies the requirements.

Since $\bar{H}^2$ give a system isomorphic to R^2, we may assume without loss of generality that $F^1 \cap P^2 = \varnothing$. The exact sequence

$$0 \to (0, \mathbf{C} \cdot 1) \to R^2 \to \bigoplus_{\theta \in P^2} \mathrm{II}_\theta^{h_\theta} \to 0,$$

where $h_\theta = H_\theta^2$, yields by Lemma 1.2 (a) an exact sequence

$$\mathrm{Ext}\Big(\bigoplus_{\theta \in P^2} \mathrm{II}_\theta^{h_\theta}, R^1 \Big) \to \mathrm{Ext}(R^2, R^1) \to \mathrm{Ext}(\mathrm{III}^1, R^1).$$

The last term vanishes, since systems of type III^1 are projective. The first term is isomorphic to $\prod_{\theta \in P^2} \mathrm{Ext}(\mathrm{II}_\theta^{h_\theta}, R^1)$. Since $F^1 \cap P^2 = \varnothing$, we have $H_\theta^1 =$

TABLE 1

$\dim \operatorname{Ext}((V^2, W^2), (V^1, W^1))$

(V^2, W^2) \ (V^1, W^1)	I^m	II_η^m	III^m	II_η^∞	R^{1a}
I^n	$\max(0, m-n-1)$	m	$n+m$	∞ (1.4)	∞ (2.1)
II_θ^n	0	$\min(n, m)\,\delta_{\theta,\eta}$	n	0 (1.1(b))	0 if $H_\theta^1 = \infty$ n if $H_\theta^1 < \infty$ (2.2)
III^n	0	0	$\max(0, n-m-1)$	0 (1.5)	0 (2.9)
$\mathrm{II}_\theta^\infty$	0 (1.1(a))	$m\delta_{\theta,\eta}$ (1.3)	∞ (2.4)	0 (1.1(c))	0 if $H_\theta^1 = \infty$ ∞ if $H_\theta^1 < \infty$ (2.3)
R^{2a}	0 (1.5)	0 (1.5)	∞ (2.7)	0 (1.5)	0 if $F^1 \subset F^2$, $F^1 \cap P^2$ finite ∞ otherwise (2.9)

[a] R^1 and R^2 are both infinite-dimensional torsion-free of rank 1.

∞ for every $\theta \in P^2$. Thus by Propositions 2.2(a), 2.3(a) all the factors $\mathrm{Ext}(\mathrm{II}_{\theta}^{h_\theta}, R^1)$ are zero. This proves the lemma. ■

The last three lemmas imply:

PROPOSITION 2.9. *Suppose that at least one of the torsion-free systems of rank* 1, R^1 *and* R^2, *is infinite dimensional. Then* $\dim \mathrm{Ext}(R^2, R^1)$ *is either zero or infinity. It is zero if and only if the following two conditions are satisfied*:

(i) *for all* $\theta \in \tilde{\mathbf{C}}$, $H_\theta^1 < \infty$ *implies* $H_\theta^2 < \infty$;
(ii) $\{\theta \in \tilde{\mathbf{C}} : H_\theta^1 < \infty$ *and* $H_\theta^2 > 0\}$ *is finite.*

Table 1 summarizes the results of the paper. It also includes the finite-dimensional case treated in [3]. The numbers in parentheses refer to the appropriate statement in the paper.

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Received 28 August 1976